

The Hertling conjecture in dimension 2

Thomas Brélivet

Dedicated to the memory of my father who died during the preparation of this manuscript.

Abstract

We consider an isolated plane curve singularity and its associated Eisenbud and Neumann diagram. We give an algorithm to compute the maximal spectral value on the diagram and we show that the singularity is topologically equivalent to another singularity for which the maximal spectral value is given by the point $(1,1)$ in the plane of the Newton polygon. From the almost additivity on the splice components of the diagram we compute the sum of the square of the spectral values. This formula with the previous result on the maximal spectral value give us the Hertling conjecture as a corollary.

Résumé

On considère une singularité isolée de courbe ainsi que son diagramme de Eisenbud et Neumann. On donne un algorithme pour calculer la valeur spectrale maximale sur le diagramme et on montre que la singularité est topologiquement équivalente à une autre telle que sa valeur spectrale maximale est donnée par le point $(1,1)$ dans le plan du polygone de Newton. De la presque additivité sur les composantes de splice du diagramme, on calcule la somme des carrés des valeurs spectrales. Cette formule ainsi que les résultats précédents sur la valeur spectrale maximale nous donne la conjecture de Hertling comme corollaire.

Contents

1	Introduction	2
2	Definition of the spectral pairs of a fibered Eisenbud and Neumann diagram	3
3	Geometric description of the spectrum	5
4	The non-degenerated and commode case	8

AMS classification (2000): 14B05, 14H20, 52C05,

Keywords: Isolated singularities, Spectrum singularity, Hertling conjecture, Newton polygon

5	Newton polygonal representation and additivity of the spectral pairs	11
6	The maximal spectral value	14
7	A formula for the variance of the spectrum, Hertling conjecture	18

1 Introduction

Some 25 years ago, Steenbrink has defined the spectrum of an isolated hypersurface singularity in [St1] and then Steenbrink himself, Varchenko [V] and others have obtained very interesting results motivated mainly by one conjecture made by Arnold, see [AGV] and [St2] for more details.

The spectrum is a collection of rational numbers between -1 and n , where $n+1$ denotes the dimension of the ambient space and it is symmetric around $(n-1)/2$.

The variance measures the distribution of these numbers with respect to the middle point and is defined by

$$V = \frac{1}{\mu} \sum_{i=1}^{\mu} \left(\alpha_i - \frac{n-1}{2} \right)^2$$

where $\alpha_1 + \dots + \alpha_{\mu}$ as an element of $\mathbb{N}^{(\mathbb{Q})}$ is the spectrum with $\alpha_1 \leq \dots \leq \alpha_{\mu}$.

It came as a great surprise when Hertling, at the Summer Institute on Singularities, Newton Institute, Cambridge 2000, proposed the following conjecture.

Conjecture 1.1 *For any isolated hypersurface singularity*

$$V \leq \frac{\alpha_{\mu} - \alpha_1}{12}.$$

This conjecture was supported at the time by the case of weighted homogeneous singularities where one has in fact an equality (see [H] for a conceptual proof involving Frobenius manifolds and [Di] for a high school proof based on some formulas in [St1]) as well as by inspection through Arnold's lists of singularities.

Soon after this, M. Saito (see [S2]) showed that Conjecture 1.1 holds for all irreducible plane curves singularities. In 2002 it has been proved by the author that Conjecture 1.1 also holds for all non-degenerated and comode plane curves singularities.

Here we prove the Conjecture 1.1 for all isolated plane curve singularities, see Corollary 7.8.

In section 2 we recall how to compute the spectral pairs of an isolated curve singularity.

Note that in every dimension the spectrum of a Newton non-degenerated singularity is known from the Newton polyhedron by Steenbrink [St1], Khovanskii and Varchenko [KV].

In section 3 we give a geometric description in terms of Newton polygons, generalizing the well known situation of the Newton Non-degenerated case.

In section 4 we recall the formulas in the non-degenerated case.

In section 5 we construct an application from the set of isolated singularities plane curve in the free group generated by the commode Newton polygons.

In section 6 we give an algorithm to compute the maximal spectral value and we prove the Theorem 6.7.

In every dimension we already know that the multiplicity of the minimal spectral value is one, see [S1]. We Thank Antoine Douai for this reference.

Finally the section 7 is the core of the proof of the conjecture and gives us an expression of $6S - \mu\alpha_\mu$ (S is the sum of the square of the spectral values and α_μ the maximal spectral value) as a linear combination with strictly negative coefficients of the determinants of the Eisenbud and Neumann diagram representing the link of f . The formula gives the Hertling conjecture as a corollary.

2 Definition of the spectral pairs of a fibered Eisenbud and Neumann diagram

In this section we define a notion of spectral pairs associated to a fibered multilink, see [SSS] and [C]. Let L be a fibered multilink and $(\Gamma, *)$ a rooted Eisenbud and Neumann diagram with non zero determinant representing L (for instance minimal). See [EN], [CP] for a complete introduction of Eisenbud and Neumann diagram and [N] for a rapid and historic introduction.

Let V the set of vertices, E the set of edges, A the set of arrows and R the set of rupture vertices (vertices such that the number of incident edges is greater or equal than 2) of Γ .

Let v be a vertex of Γ . Cut edges joining v and rupture vertices. Replace the edges by arrows with multiplicities such that the multiplicity m_v of v doesn't change (see Figure 1). We define $m_i = 0$ for $i = k + 1, \dots, n_v$, β_j for $j = 1, \dots, n_v$ such that

$$\beta_j \alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_{n_v} \equiv 1 \pmod{\alpha_j}$$

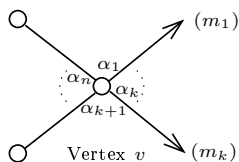


Figure 1: Neighborhood of v

and

$$s_{v,w} = s_{v,j} = \frac{m_j - \beta_j m_v}{\alpha_j}, \text{ where } w \text{ is the end of the } j^{\text{th}} \text{ edge.}$$

- (1) If v is not the root then let $p(v)$ be the predecessor of v given by the first vertex met in the path from v to $*$ (it is unic because Γ is a tree),
- (2) m_v the multiplicity of the vertex v , it is the sum over the arrows of the product of all edge weights adjacent to the path from v to the arrows,
- (3) $d_v = \gcd(m_v, s_{v,p(v)})$,
- (4) $r_v = \gcd(m_v, s_{v,p(v)}, j = 1, \dots, n_v)$.

Then we define some elements of $\mathbb{Z}^{(\mathbb{Q} \times \mathbb{Z})}$ by

$$a_v = \sum_{\substack{0 < s < m_v \\ m_v \nmid s r_v}} \left(-1 + \sum_{i=1}^{n_v} \left\{ \frac{s s_{v,i}}{m_v} \right\} \right) \left[\left(\frac{s}{m_v} - 1, 1 \right) + \left(1 - \frac{s}{m_v}, 1 \right) \right], \quad v \in V \cup \{*\},$$

$$b_v = \sum_{0 < s < r_v} \left[\left(-\frac{s}{r_v}, 2 \right) + \left(\frac{s}{r_v}, 0 \right) \right], \quad v \in V \cup \{*\},$$

$$c_v = \sum_{0 < s < d_v} \left[\left(-\frac{s}{d_v}, 2 \right) + \left(\frac{s}{d_v}, 0 \right) \right], \quad v \in V,$$

$$c'_v = \sum_{0 < s < d_v} \left(-\frac{s}{d_v}, 2 \right), \quad v \in A,$$

where $\{x\}$ for $x \in \mathbb{R}$ means the fractional part of x .

Definition 2.1 *The spectral pairs of L are defined by*

$$\text{Spp}(L) = \sum_{v \in R} a_v + \sum_{v \in R \setminus \{*\}} (c_v - b_v) - b_* + \sum_{v \in A} c'_v + (|A| - 1)(0, 1).$$

and the spectrum of L is defined by the projection on the first factor of the spectral pairs and is denoted by $\text{Sp}(L)$.

Remark 2.2 The definition of the spectral pairs of a fibered multilink is an invariant of the topology of the complementary of the link and is independant of the choice of the root. It is also independant of the diagram if we not permit zero determinants. The spectrum is independant of the choice of the root and the diagram (even if we accept zero determinants).

Theorem 2.3 ([SSS]) *Let f be an isolated plane curve singularity and L_f the link associated to f . Then $\text{Spp}(L_f) = \text{Spp}(f)$.*

The following proposition shows that Spp is almost additive. The section 5 will explain that through a factorisation Spp is additive.

Proposition 2.4 ([SSS]) *Suppose that the fibered multilink L is the result of splicing the fibered multilinks L_1 and L_2 along components of multilink multiplicities m_1 and m_2 . Let $d = \gcd(m_1, m_2)$ then*

$$\text{Spp}(L) = \text{Spp}(L_1) + \text{Spp}(L_2) - (0, 1) + \sum_{s=1}^{d-1} \left[\left(\frac{s}{d}, 0 \right) - \left(-\frac{s}{d}, 2 \right) \right].$$

We denote by $\alpha_1, \dots, \alpha_\mu$ the spectral values, where μ is the number of spectral values counted with multiplicity and $\alpha_1 \leq \dots \leq \alpha_\mu$.

3 Geometric description of the spectrum

Let us consider $\Gamma(m, n, p, q, \ell_1, \dots, \ell_a)$ the (minimal) diagram defined by the Figure 2 where m, n are non negative integers, (p, q) are coprime positive integers, ℓ_1, \dots, ℓ_a are positive numbers, $\ell = \ell_1 + \dots + \ell_a$, with $m - p\ell > 0$.

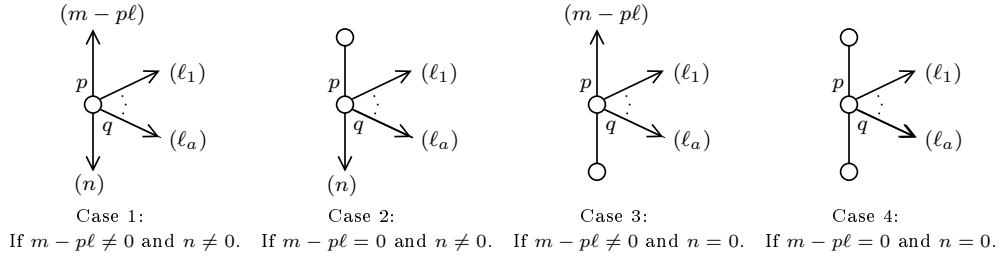


Figure 2: Parallelogram Diagrams

Diagrams of this type are the bricks of the Eisenbud and Neumann diagrams.

As we have seen in the Remark 2.2 the spectrum of $\Gamma(m, n, p, q, \ell_1, \dots, \ell_a)$ and of the Figure 3 are equals.

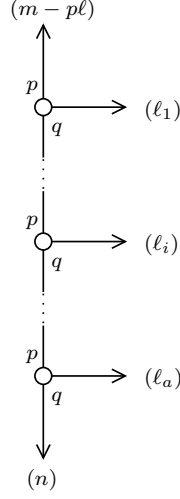


Figure 3: Separated Arrows

So from now we suppose that $a = 1$ and $\ell_1 = \ell$ so the diagram is $\Gamma(m, n, p, q, \ell)$ and we compare its spectrum with the spectrum of $\Gamma(p, q, m, n, \underbrace{1, \dots, 1}_{\ell \text{ times}})$.

Chose u and v such that $up + vq = 1$. And note $r = \gcd(\ell, m, n)$, $d = \gcd(\ell, N)$, $d_0 = \gcd(m - p\ell, n + q\ell)$ and $d_\infty = \gcd(m, n)$. Then we have

$$\begin{aligned} \text{Spp}(\Gamma(m, n, p, q, \ell)) = & \sum_{\substack{0 < s < N \\ N \nmid sr}} \left(-1 + \left\{ \frac{s\ell}{N} \right\} + \left\{ \frac{s(um - vn - \ell)}{N} \right\} + \left\{ \frac{s(vn - um)}{N} \right\} \right) \left(\pm \left(\frac{s}{N} - 1 \right), 1 \right) \\ & - \sum_{0 < s < r} \left[\left(-\frac{s}{r}, 2 \right) + \left(\frac{s}{r}, 0 \right) \right] + \sum_{0 < s < d} \left(-\frac{s}{d}, 2 \right) + B(\Gamma(m, n, p, q, \ell)) + 2(0, 1), \end{aligned}$$

and

$$\begin{aligned} \text{Spp}(\Gamma(p, q, m, n, \underbrace{1, \dots, 1}_{\ell \text{ times}})) = & \sum_{0 < s < N} \left(-1 + \frac{s\ell}{N} + \left\{ \frac{s(um - vn - \ell)}{N} \right\} + \left\{ \frac{s(vn - um)}{N} \right\} \right) \left(\pm \left(\frac{s}{N} - 1 \right), 1 \right) \\ & + B(\Gamma(p, q, m, n, \underbrace{1, \dots, 1}_{\ell \text{ times}})) + (t + 1)(0, 1), \end{aligned}$$

where

$$B(\Gamma(m, n, p, q, \ell)) = \begin{cases} \sum_{s=1}^{d_0-1} \left(-\frac{s}{d_0}, 2\right) + \sum_{s=1}^{d_\infty-1} \left(-\frac{s}{d_1}, 2\right) & \text{Case 1,} \\ \sum_{s=1}^{d_1-1} \left(-\frac{s}{d_\infty}, 2\right) & \text{Case 2,} \\ \sum_{s=1}^{d_0-1} \left(-\frac{s}{d_0}, 2\right) & \text{Case 3,} \\ 0 & \text{Case 4,} \end{cases}$$

and $B(\Gamma(p, q, m, n, \underbrace{1, \dots, 1}_{\ell \text{ times}})) = B(\Gamma(m, n, p, q, \ell))$.

As we will see the following Lemma is very usefull to understand the geometry of the spectrum.

Lemma 3.1 *Let x and y be real numbers such that $x < y$. Then we have*

$$\text{card}([x, y[\cap \mathbb{Z}) = y - x + \{x\} + \{-y\} - 1.$$

Let ϕ the linear map defined by

$$\begin{aligned} \phi : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \frac{qx+py}{qm+pn}. \end{aligned}$$

We then have a generalisation relative to a cone.

Lemma 3.2 *Let s in \mathbb{Z} , $(m_0, n_0), (m_1, n_1) \in (\mathbb{N}^*)^2$, \mathbb{Q} linearly independants, K the open cone in \mathbb{R}_+^2 generated by (m_0, n_0) and (m_1, n_1) . Then we have*

$$\begin{aligned} \text{card} \left(K \cap \phi^{-1} \left(\frac{s}{p_1 n_1 + q_1 m_1} \right) \cap \mathbb{N}^2 \right) = \\ -1 + \frac{k_1}{p_1 n_1 + q_1 m_1} s + \left\{ \frac{u m_0 - v n_0}{p_1 n_1 + q_1 m_1} s \right\} + \left\{ \frac{v n_1 - u m_1}{p_1 n_1 + q_1 m_1} s \right\}. \end{aligned}$$

Now we are ready to write a geometric interpretation of the spectrum in the non-degenerated case.

Proposition 3.3

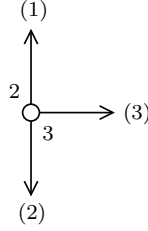
$$\begin{aligned} \text{Spp}(\Gamma(p, q, m, n, \underbrace{1, \dots, 1}_{\ell \text{ times}})) = \\ \sum_{(m, n) \in P} (1 - \phi(m, n), 1) + B(\Gamma(p, q, m, n, \underbrace{1, \dots, 1}_{\ell \text{ times}})) + (t+1)(0, 1), \end{aligned}$$

where P is the open parallelogram generated by (m, n) and $(m - pl, n + ql)$.

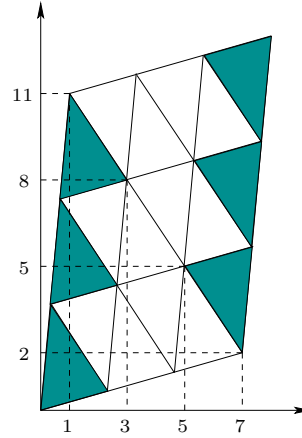
Now the following formula permits us to understand the geometry in the non degenerated case.

$$\begin{aligned}
\text{Spp}(\Gamma(m, n, p, q, \ell)) &= \text{Spp}(\Gamma(p, q, m, n, \underbrace{1, \dots, 1}_{\ell \text{ times}})) \\
&- \sum_{0 < s < N} \left(\frac{s\ell}{N} - \left\{ \frac{s\ell}{N} \right\} \right) \left(\pm \left(\frac{s}{N} - 1 \right), 1 \right) + \sum_{\substack{0 < s < N \\ N/sr}} \left(\pm \left(\frac{s}{N} - 1 \right), 1 \right) \\
&- \sum_{0 < s < r} \left[\left(-\frac{s}{r}, 2 \right) + \left(\frac{s}{r}, 0 \right) \right] + \sum_{0 < s < d} \left(-\frac{s}{d}, 2 \right) - (t-1)(0, 1).
\end{aligned}$$

To avoid long explanation we give an example of geometric representation of $\Gamma(7, 2, 2, 3, 3)$ in the Figure 4 in order to understand the geometry of the spectrum when it is degenerated with respect to the Newton polygon.



(a) Eisenbud and Neumann diagram of $\Gamma(7, 2, 2, 3, 3)$



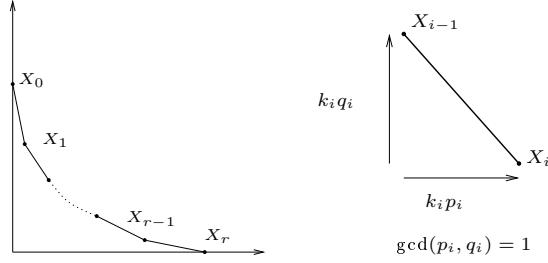
(b) Geometric representation of $\Gamma(7, 2, 2, 3, 3)$

Figure 4: Example of geometric representation of the spectrum

4 The non-degenerated and commode case

Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be an isolated singularity of curve. We suppose that f is Newton non-degenerated and commode.

The Newton polygon is given by the Figure 5(a)



(a) Local case

(b) One face

Figure 5: Newton polygon

and the Eisenbud and Neumann diagram is given by the Figure 6.

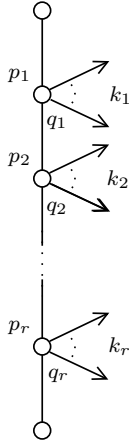


Figure 6: Eisenbud and Neumann diagram

The correspondance between $(X_i = (m_i, n_i))_{0 \leq i \leq r}$ (r is the number of faces of the Newton polygon of f) and $(p_i, q_i, k_i)_{1 \leq i \leq r}$ is given by

$$\begin{aligned} m_i &= k_1 p_1 + \cdots + k_i p_i, \\ n_i &= k_{i+1} q_{i+1} + \cdots + k_r q_r, \end{aligned}$$

for $0 \leq i \leq r$ and

$$\begin{aligned} k_i &= \gcd(m_i - m_{i-1}, n_{i-1} - n_i), \\ p_i &= (m_i - m_{i-1})/k_i, \\ q_i &= (n_{i-1} - n_i)/k_i. \end{aligned}$$

for $1 \leq i \leq r$.

It is usefull to introduce two new points $X_{-1} = (1, 1)$, $X_0 = (1, 1)$ and the following notations:

$$d_i = \gcd(m_i, n_i), \quad 1 \leq i \leq r,$$

$$A_{i,j} = m_i n_j - m_j n_i, \quad -1 \leq i, j \leq r+1, \text{ in particular } A_{i,i-1} = k_i(p_i n_i + q_i m_i),$$

$$P_i = \{\lambda_1 X_{i-1} + \lambda_2 X_i : 0 < \lambda_i < 1, \quad i = 1, 2\}, \quad 1 \leq i \leq r,$$

$$L_i =]0, 2X_i[, \quad 1 \leq i \leq r-1$$

$$\phi_i \text{ the linear map which take the value 1 on } X_{i-1} \text{ and } X_i, \text{ for } 1 \leq i \leq r,$$

$$\Delta_i = p_{i+1} q_i - p_i q_{i+1}, \quad 1 \leq i \leq r.$$

Let call $\Gamma_{ND}(p_1, q_1, k_1; \dots; p_r, q_r, k_r)$ such a diagram.

Remark 4.1 Due to the local situation we know that $\Delta_i > 0$. We can permit $k_i = 0$ by putting $\Gamma_{ND}(p_1, q_1, k_1; \dots; p_r, q_r, k_r) = \Gamma_{ND}(p_1, q_1, k_1; \dots; \hat{p}_i, \hat{q}_i, \hat{k}_i; \dots; p_r, q_r, k_r)$ where the hat means that the term is omitted.

From the previous sections we have the following well known equalities.

Proposition 4.2 *The spectrum of f is*

$$Sp(f) = \sum_{i=1}^r \sum_{(m,n) \in P_i \cap \mathbb{N}^2} (1 - \phi_i(m, n)) + \sum_{i=1}^{r-1} \sum_{(m,n) \in L_i \cap \mathbb{N}^2} (1 - \phi_i(m, n)).$$

and the Milnor number of f is

$$\mu(f) = A_{0,-1} + A_{1,0} + \dots + A_{r+1,r} + 1.$$

In [B1] it has been proved that we have the following theorem.

Theorem 4.3 *We have*

$$6S - \mu \alpha_\mu = - \sum_{i=1}^{r-1} E_i \Delta_i,$$

where

$$\alpha_\mu = 1 - \phi_{i_0}(1, 1)$$

is the maximal spectral value with i_0 such that $(1, 1)$ is in the parallelogram generated by X_{i_0-1}, X_{i_0} and

$$E_i = \begin{cases} \left(\sum_{k=-1}^{i_0-1} A_{k+1,k} (n_i - m_i) + d_i^2 - m_i \right) \frac{k_i k_{i+1}}{A_{i,i-1} A_{i+1,i}}, & \text{if } 1 \leq i < i_0, \\ \left(\sum_{k=i_0}^r A_{k+1,k} (m_i - n_i) + d_i^2 - n_i \right) \frac{k_i k_{i+1}}{A_{i,i-1} A_{i+1,i}}, & \text{if } i_0 \leq i \leq r. \end{cases}$$

Remark 4.4 The previous formula is true for every $i \in \{1, \dots, r\}$, we use it for i_0 in order to prove the Hertling conjecture.

The quantities E_i are strictly positive so we deduce the following corollary.

Corollary 4.5 *If f is a germ Newton non-degenerated and commode then the conjecture of Hertling is true for f . Moreover, we have an equality if and only if f is a positive deformation of a quasi-homogeneous polynomial defining an isolated singularity.*

Notation 4.6 To simplify the computations we need to introduce:

$$F_i = \begin{cases} \sum_{k=-1}^{i_0-1} A_{k+1,k}(n_i - m_i) + d_i^2 - m_i & \text{if } 1 \leq i < i_0, \\ \sum_{k=i_0}^r A_{k+1,k}(m_i - n_i) + d_i^2 - n_i & \text{if } i_0 \leq i \leq r, \end{cases}$$

and

$$C_i = \frac{A_{i,i-1} + A_{i+1,i} - A_{i+1,i-1}}{A_{i,i-1}A_{i+1,i}} = -\frac{1}{(p_i n_i + q_i m_i)(p_{i+1} n_{i+1} + q_{i+1} m_{i+1})} \Delta_i.$$

The formula becomes

$$6S - \mu\alpha_\mu = \sum_{i=1}^{r-1} F_i C_i.$$

Remark 4.7 We have the following usefull equality

$$\frac{(n_i - m_i)\Delta_i}{(p_i n_i + q_i m_i)(p_{i+1} n_{i+1} + q_{i+1} m_{i+1})} = \phi_i(1, 1) - \phi_{i+1}(1, 1).$$

5 Newton polygonal representation and additivity of the spectral pairs

Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be an isolated singularity of curve. From the Newton Puiseux algorithm, we can compute the Eisenbud and Neumann diagram of an isolated plane curve singularity. This algorithm follows a tree called here the polygon tree \mathbf{P}_f of f , 0 being the root of \mathbf{P}_f and gives a splice diagram Γ_f . Each vertex of this tree correspond a polygon.

In the following, without loss of generality we can suppose that we can use the representation of a Eisenbud and Neumann diagram given by the Figure 7 representing f .

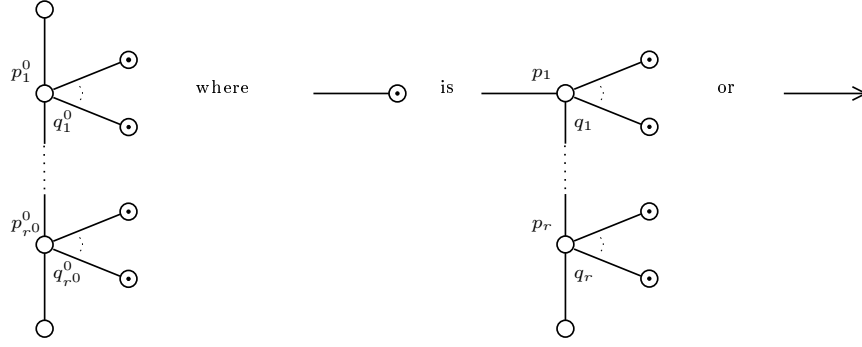


Figure 7: A general Eisenbud and Neumann diagram

Definition 5.1 Let \mathcal{P}^a be the \mathbb{Z} -free module generated by the abstract Newton polygons $\Gamma_{ND}(p_1, q_1, k_1; \dots; p_r, q_r, k_r)$ where k_1, \dots, k_r are positive integers and $(p_1, q_1), \dots, (p_r, q_r)$ are positive coprime integers.

Let $\text{Sppa} : \mathcal{P}^a \longrightarrow \mathbb{Z}^{(\mathbb{Q})}$ the morphism such that $\text{Sppa}(\Gamma_{ND}(p_1, q_1, k_1; \dots; p_r, q_r, k_r)) = \text{Spp}(\Gamma_{ND}(p_1, q_1, k_1; \dots; p_r, q_r, k_r))$.

Cut the horizontal edges (those are not vertical) and put arrows with multiplicities such that the multiplicities of each vertex don't change. See Figures 8.

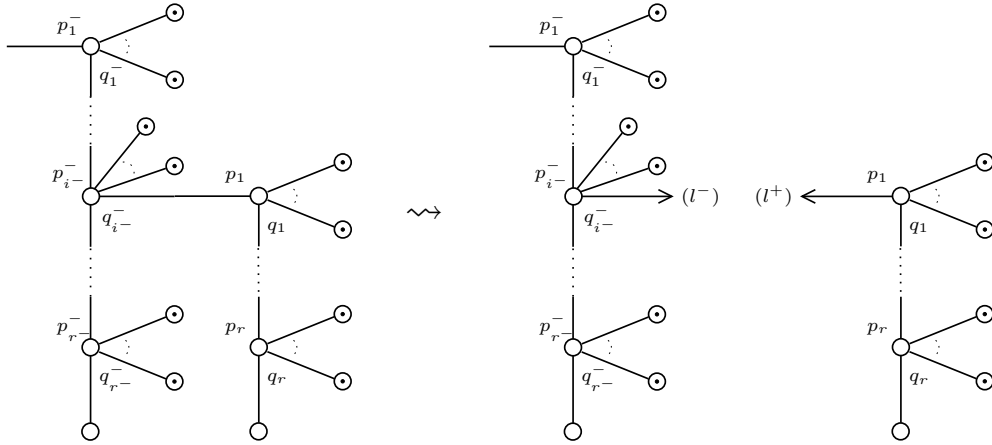


Figure 8: Two consecutive vertical parts of the Eisenbud and Neumann diagram

Then replace horizontal arrows with multiplicity l by l arrows of multiplicity 1 and subtract by a new diagram in order to obtain polygonal diagrams. See Figure 9.

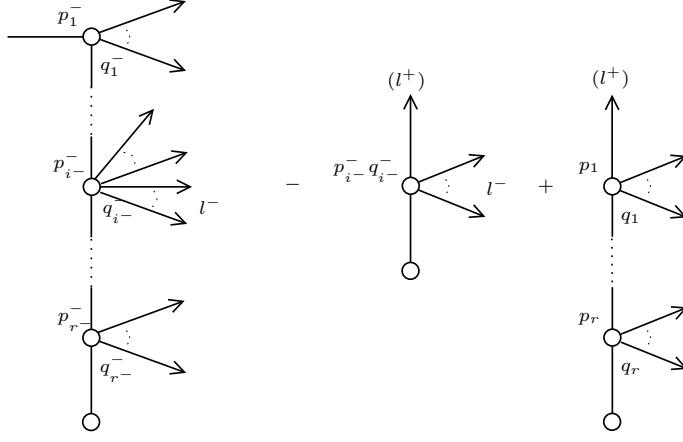


Figure 9: Cutted horizontal edges

If necessary ($l^+ \neq 0$) complete the diagrams in order to have commode polygonal diagrams. See Figure 10 for the last two parts.

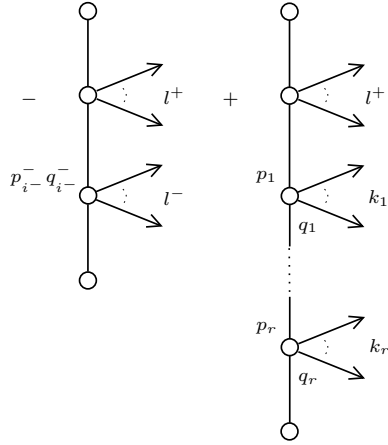


Figure 10: Vertical parts in \mathcal{P}^a

In this way we have construct an element of \mathcal{P}^a :

$$\Psi(\Gamma_f) = \Psi_0(\Gamma_f) + \sum_{w \in \mathbf{P}_f} \Psi_w(\Gamma_f)$$

where $\Psi_0(\Gamma_f) = \Gamma_{ND}(p_1^0, q_1^0, k_1^0; \dots; p_r^0, q_r^0, k_r^0)$ is the Newton polygon of the germ,

$$\Psi_w(\Gamma_f) = \Psi_w^+(\Gamma_f) - \Psi_w^-(\Gamma_f),$$

$$\Psi_w^+(\Gamma_f) = \Gamma(1, 1, l^+; p_1, q_1, k_1; \dots; p_r, q_r, k_r)$$

and

$$\Psi_w^-(\Gamma_f) = \Gamma(1, 1, l^+; p_i^- q_i^-, 1, l^-).$$

Theorem 5.2 *From Ψ , we get an equivalence relation \sim on \mathcal{P}^a and a map*

$$\{f \in \mathbb{C}\{X, Y\} : f \text{ defines an isolated singularity}\} \rightarrow \mathcal{P}^a / \sim$$

such that the composition with the morphism Sppa gives the spectral pairs of f .

Question 5.3 *What is $\ker(\text{Sppa})$? Do we have such a factorisation in higher dimensions?*

6 The maximal spectral value

Recall that to $\Gamma(m, n, p, q, \ell_1, \dots, \ell_a)$ we have associated

$$\begin{aligned} \phi : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x, y) &\mapsto \frac{qx+py}{qm+pn}. \end{aligned}$$

In the non-degenerated case the maximal spectral value is given by $1 - \phi(1, 1)$ where the ϕ corresponds to the parallelogram wich contains the point $(1, 1)$ so it is natural to put the following definition.

Definition 6.1 *We call $1 - \phi(1, 1)$ the virtual spectral value associated to $\Gamma(m, n, p, q, \ell_1, \dots, \ell_a)$.*

We get a majoration of the spectral values (those of multiplicity non zero).

Proposition 6.2 *We have*

$$1 - \phi(1, 1) \geq \alpha,$$

for all spectral value α of $\Gamma_P(m, n, p, q, \ell_1, \dots, \ell_a)$. The virtual spectral value $1 - \phi(1, 1)$ is a spectral value if and only if $(1, 1) \in \{\lambda_0(m - p\ell, n + q\ell) + \lambda_1(m, n) : 0 \leq \lambda_0, \lambda_1 < 1\}$. If it is not the case, then $\phi(1, 1)$ is a spectral value.

We will show that this majoration is enough for our computations.

Now we can define the virtual spectral value associated to a vertex v of the Eisenbud and Neumann diagram of the germ f . Consider a vertex v of the Eisenbud and Neumann diagram Γ_f and the diagram associated to the vertex v obtained by cutting all the edges around v . Then we get a diagram of the type $\Gamma(m, n, p, q, \ell_1, \dots, \ell_a)$.

Definition 6.3 *The virtual maximal spectral value of the vertex v is the virtual spectral value associated to the previous diagram.*

To study the maximal spectral value of Γ_f it is usefull to study the variation in the diagram of the virtual spectral values.

Consider the part of the Eisenbud and Neumann diagram given by the Figure 11.

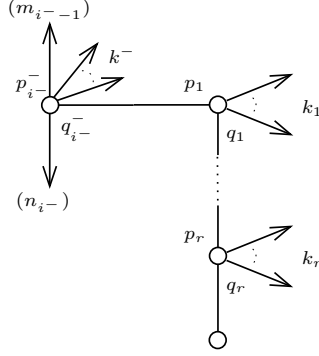


Figure 11: Attached vertical part

Let

$$\ell^+ = p_{i-}^{-} n_{i-} + q_{i-}^{-} m_{i-1} + p_{i-}^{-} q_{i-}^{-} k^{-}$$

and

$$\ell^- = q_1 k_1 + \cdots + q_r k_r.$$

Proposition 6.4 *Consider an edge of the diagram of the Figure 11. Cut the edge and get two arrows with multiplicities. Then*

- 1- *if the edge is vertical the arrow with maximal multiplicity gives the direction where we have to go to get the virtual maximal spectral value,*
- 2- *if the edge is horizontal and $p_{i-}^{-}, q_{i-}^{-} > 1$ or $\ell^+ > \ell^-$ then the virtual maximal spectral value is given by the vertex of index i^- .*

Proof. The first assertion is given for the vertical part (non-degenerate case) of the diagram by the remark 4.7.

The second assertion is given by the following computation.

Let

$$\alpha_{i-}^{-} = 1 - \frac{p_{i-}^{-} + q_{i-}^{-}}{\ell^+ + p_{i-}^{-} q_{i-}^{-} \ell^-}$$

and

$$\alpha_j = 1 - \frac{p_j + q_j}{q_j(\ell^+ + p_1 k_1 + \cdots + p_j k_j) + p_j(q_{j+1} k_{j+1} + \cdots + q_r k_r)}.$$

We have to show that $\alpha_{i-}^{-} > \alpha_j$ for j from 1 to r .

We have (see section 7 for the definition of Δ_{jb})

$$\begin{aligned}
q_j(\ell^+ + p_1 k_1 + \cdots + p_j k_j) + p_j(q_{j+1} k_{j+1} + \cdots + q_r k_r) \\
&= q_j(\ell^+ + p_1 k_1 + \cdots + p_j k_j) + p_j(\ell^- - q_1 k_1 - \cdots - q_j k_j) \\
&= p_j \ell^- + q_j \ell^+ + \sum_{b=1}^{j-1} (p_b q_j - p_j q_b) k_b \\
&= p_j \ell^- + q_j \ell^+ - \sum_{b=1}^{j-1} \Delta_{jb} k_b
\end{aligned}$$

the numerator of $\alpha_{i^-}^- - \alpha_j$ is

$$[p_j + q_j - q_j(p_{i^-}^- + q_{i^-}^-)]\ell^+ + [p_{i^-}^- q_{i^-}^- (p_j + q_j) - p_j(p_{i^-}^- + q_{i^-}^-)]\ell^- + (p_{i^-}^- + q_{i^-}^-) \sum_{b=1}^{j-1} \Delta_{jb} k_b,$$

and

$$\begin{aligned}
p_j + q_j - q_j(p_{i^-}^- + q_{i^-}^-) &= q_j(p_{i^-}^- - 1)(q_{i^-}^- - 1) + (p_j - q_j p_{i^-}^- q_{i^-}^-), \\
p_{i^-}^- q_{i^-}^- (p_j + q_j) - p_j(p_{i^-}^- + q_{i^-}^-) &= p_j(p_{i^-}^- - 1)(q_{i^-}^- - 1) - \Delta_0 = p_j((p_{i^-}^- - 1)(q_{i^-}^- - 1) - 1) + p_{i^-}^- q_{i^-}^- q_j.
\end{aligned}$$

The numerator can be rewrite as

$$(q_j \ell^+ + p_j \ell^-)(p_{i^-}^- - 1)(q_{i^-}^- - 1) + (\ell^+ - \ell^-) \Delta_0 + (p_{i^-}^- + q_{i^-}^-) \sum_{b=1}^{j-1} \Delta_{jb} k_b$$

so it is positive if $p_{i^-}^-, q_{i^-}^- > 1$ or $\ell^+ > \ell^-$.

□

Condition (H1): For each vertex of the diagram all the horizontal edges attached to a vertex except perhaps one are such that $\ell^- \geq \ell^+$.

Suppose that there is two horizontal edges e_1, e_2 attached (from the right) to the vertex $v \in V$ with decoration (p, q) such that $\ell_1^- \geq \ell_1^+$ and $\ell_2^- \geq \ell_2^+$. By construction we have $\ell_1^+ = pq\ell_2^- + a_1$, $\ell_2^+ = pq\ell_1^- + a_2$, with $a_1, a_2 \geq 0$. Then $\ell_1^+ \geq \ell_2^-$ and $\ell_2^+ \geq \ell_1^-$. From this we deduce that $\ell_1^+ = \ell_1^- = \ell_2^+ = \ell_2^-$, $p = q = 1$ and the diagram is in the form given by the Figure 12.

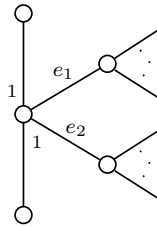


Figure 12: Neighborhood of v

We can eliminate the two vertices of valence 1 and the edge e_1 and e_2 become vertical. These operations don't change the spectrum and by this way we can suppose that we are in the situation of the condition (H1).

Now we can look for the maximal virtual spectral value of Γ_f .

The point $(1, 1)$ give us the vertex we must consider in the first Newton polygon (corresponding to the vertex $0 \in \mathbf{P}_f$). Let i_0^0 be the index of this vertex. Consider the horizontal edges attached to it. There is at most one edge such that $\ell^- \geq \ell^+$. If there is no such edge then the algorithm stops. If not consider the vertex attached to the right of the edge. It gives a new polygon (so a new vertex of \mathbf{P}_f) and we consider the new vertex corresponding given by the point $(1, 1)$. If $p_{i_0^0}$ and $q_{i_0^0}$ are different to 1 then we continue the algorithm with the new vertex instead of the vertex of index i_0 . If not the we exchange the horizontal edge with the vertical one of decoration 1 and we continue the algorithm.

If we do again the algorithm then we get a sequence of vertices (and the tree is not modified!) $v_{i_0^0}, \dots, v_{i_0^n}$. Let $\alpha_{i_0^0}, \dots, \alpha_{i_0^n}$ the corresponding virtual spectral values and $(p_{i_0^0}, q_{i_0^0}), \dots, (p_{i_0^n}, q_{i_0^n})$ the decorations attached to these vertices. The algorithm tells us that the decorations $p_{i_0^0}, q_{i_0^0}, \dots, p_{i_0^n}, q_{i_0^n}$ are all different to 1 and $\alpha_{i_0^0} > \dots > \alpha_{i_0^n}$ (case 2 of the proposition). Take now any vertex v of the tree Γ_f with virtual spectral value α_v and consider the geodesic from v to $v_{i_0^0}$. Until we have not reach a vertex of the sequence $v_{i_0^0}, \dots, v_{i_0^n}$, in order to find the maximal virtual spectral value we follow the direction given by the arrow of maximal multiplicity. From the Proposition 6.4 the virtual spectral values are growing. Call $v_{i_0^0}^k$ the first vertex of the sequence we met. To conclude, we have $\alpha_v \leq \alpha_{i_0^0}^k \leq \alpha_{i_0^0}$.

Proposition 6.5 *The virtual spectral value*

$$\alpha_{i_0^0}$$

is the maximal virtual spectral value.

Proposition 6.6 *The maximal virtual spectral value is the maximal spectral value.*

Proof. We just have to check that the virtual spectral value $\alpha_{i_0^0}$ is a spectral value.

From the choice of i_0^0 the value $\alpha_{i_0^0}$ has multiplicity 1 in

$$\text{Sp}(\Gamma(m_{i_0^0+1}, n_{i_0^0}, p_{i_0^0}, q_{i_0^0}, k_{i_0^0})).$$

The spectral values of $\text{Sp}(\Gamma(\ell_{i_0^0}^+ + p_{i_0^0} q_{i_0^0} \ell_{i_0^0}^-, 0, p_{i_0^0} q_{i_0^0}, 1, \ell_{i_0^0}^-))$ are of the form

$$1 - \frac{x + p_{i_0^0} q_{i_0^0} y}{\ell_{i_0^0}^+ + p_{i_0^0} q_{i_0^0} \ell_{i_0^0}^-}, \quad x, y \geq 1$$

and from the fact that $p_{i_0^0} > 1, q_{i_0^0} > 1$ it is strictly greater than

$$\alpha_{i_0^0} = 1 - \frac{p_{i_0^0} + q_{i_0^0}}{p_{i_0^0} n_{i_0^0} + q_{i_0^0} m_{i_0^0}} = 1 - \frac{p_{i_0^0} + q_{i_0^0}}{\ell_{i_0^0}^+ + p_{i_0^0} q_{i_0^0} \ell_{i_0^0}^-}.$$

□

Theorem 6.7 *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be an analytic germ defining an isolated singularity of curve. Then there exists a germ $g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ topologically equivalent to f such that the maximal spectral value is given by the point $(1, 1)$ in the plane of the Newton polygon of g . Furthermore the multiplicity of the maximal (or minimal) spectral value is one.*

Remark 6.8 In general the germ g is not Newton non-degenerated.

7 A formula for the variance of the spectrum, Hertling conjecture

From the previous section we can suppose that the maximal spectral value of f is given by the Newton polygon.

We have seen that we have a decomposition of the splice diagram as an abstract sum of polygons, $\Psi(\Gamma_f) = \Psi_0(\Gamma_f) + \sum_{w \in \mathbf{P}_f} (\Psi_w^+(\Gamma_f) - \Psi_w^-(\Gamma_f))$.

Let $w \in \mathbf{P}_f$ different of the root 0 and

μ_0 the Milnor number of $\Psi_0(\Gamma_f)$,

μ_w^+ the Milnor number of $\Psi_w^+(\Gamma_f)$,

μ_w^- the Milnor number of $\Psi_w^-(\Gamma_f)$,

$\mu_w = \mu_w^+ - \mu_w^-$,

S_w^+ the sum of the squares of the spectral values of $\Psi_w^+(\Gamma_f)$,

S_w^- the sum of the squares of the spectral values of $\Psi_w^-(\Gamma_f)$,

$S_w = S_w^+ - S_w^-$,

α_0 the maximal spectral value of $\Psi_0(\Gamma_f)$ (and of Γ_f),

$\ell^- = q_1 k_1 + \cdots + q_r k_r$, and ℓ^+ is the multiplicity of the left arrow,

$\alpha_w^+ = 1 - \frac{p_1 + q_1}{p_1 \ell^- + q_1 \ell^+}$,

$\alpha_w^- = 1 - \frac{2}{\ell^+ + \ell^-}$,

$p_0 = p_{i^-} q_{i^-}$, $q_0 = 1$, $\alpha_0^- = 1 - \frac{p_0 + q_0}{p_0 \ell^- + q_0 \ell^+}$,

$\Delta_{i,j} = p_i q_j - p_j q_i$, for $0 \leq i, j \leq r$,

$$\Delta_i = p_{i+1}q_i - p_iq_{i+1}, 0 \leq i \leq r-1.$$

From the additivity of the Milnor number and the sum of the squares of the spectral values we get the following lemma.

Lemma 7.1

$$6S - \mu\alpha_\mu = 6S_0 - \mu_0\alpha_0 + \sum_{w \in \mathbf{P}_f} [(6S_w^+ - \mu_w^+\alpha_w^-) - (6S_w^- - \mu_w^-\alpha_w^-) + (\mu_w^+ - \mu_w^-)(\alpha_w^- - \alpha_0)].$$

We already know from the Newton and non-degenerated case that $6S_0 - \mu_0\alpha_0$ is a linear combination with negative coefficients of the determinants of the Eisenbud and Neumann diagram. We will show that if we fix a vertex w of \mathbf{P}_f , then $(6S_w^+ - \mu_w^+\alpha_w^-) - (6S_w^- - \mu_w^-\alpha_w^-)$ and $\mu_w^+ - \mu_w^-$ are linear combination of the determinant corresponding to the vertex w . Finally we will show that $6S - \mu\alpha_\mu$ is a linear combination of the determinants with negative coefficients. From this result we deduce immediately the Hertling conjecture.

From the non-degenerated and commode case we have

$$6S^+ - \mu^+\alpha^- = F_0^+C_0^+ + \cdots + F_{r-1}^+C_{r-1}^+,$$

$$6S^- - \mu^-\alpha^- = F_0^-C_0^-,$$

where we have forgotten w to simplify notations.

Suppose that $l^+ \geq l^-$ then we know that the signs of the F_i are positive. In the case of $l^- > l^+$, the same type of computation will work.

Lemma 7.2

$$F_0^+C_0^+ - F_0^-C_0^- = [d^2 - \ell^- + (\ell^+ + p_i^-q_i^-\ell^-)(\ell^+ - \ell^-)(\ell^- - 1)](C_0^+ - C_0^-) + (\mu^+ - \mu^-)C_0^+(\ell^+ - \ell^-).$$

Proof. From the section 4 we have:

$$\begin{aligned} F_0^+ &= (A_{2,1}^+ + A_{3,2}^+ + \cdots + A_{r+2,r+1}^+)(\ell^+ - \ell^-) + d^2 - \ell^- \\ &= (\mu^+ - (\ell^+ - 1)(\ell^+ + \ell^-) - 1)(\ell^+ - \ell^-) + d^2 - \ell^- \end{aligned}$$

and

$$\begin{aligned} F_0^- &= (A_{2,1}^- + A_{3,2}^-)(m_1^- - n_1^-) + d^2 - \ell^- \\ &= (\mu^- - (\ell^+ - 1)(\ell^+ + \ell^-) - 1)(\ell^+ - \ell^-) + d^2 - \ell^-. \end{aligned}$$

So

$$F_0^+C_0^+ - F_0^-C_0^- = [d^2 - \ell^- - ((\ell^+ - 1)(\ell^- + \ell^+) + 1)(\ell^+ - \ell^-) + \mu^-(\ell^+ - \ell^-)](C_0^+ - C_0^-) + (\mu^+ - \mu^-)C_0^+(\ell^+ - \ell^-).$$

□

Lemma 7.3

$$C_0^+ - C_0^- = \frac{-\Delta_0}{(q_1\ell^+ + p_1\ell^-)(\ell^+ + p_{i^-}q_{i^-}\ell^-)}.$$

Proof. From the definitions we have:

$$C_0^+ = \frac{q_1 - p_1}{(\ell^+ + \ell^-)(q_1\ell^+ + p_1\ell^-)}$$

and

$$C_0^- = \frac{1 - p_{i^-}q_{i^-}}{(\ell^+ + \ell^-)(\ell^+ + p_{i^-}q_{i^-}\ell^-)}.$$

□

Lemma 7.4

$$\mu^+ - \mu^- = \sum_{t=0}^{r-1} \frac{n_t(n_t - 1)}{q_t q_{t+1}} \Delta_t.$$

where $n_0 = \ell^-$.

Proof. From the definitions we have:

$$\begin{aligned} \mu^+ &= (\ell^+ + \ell^-)(\ell^+ - 1) + \sum_{i=1}^r [p_i(q_i k_i + \cdots + q_r k_r) + q_i(\ell^+ + p_1 k_1 + \cdots + p_{i-1} k_{i-1})] k_i \\ &\quad - (\ell^+ + p_1 k_1 + \cdots + p_r k_r) + 1. \end{aligned}$$

and

$$\mu^- = (\ell^+ + \ell^-)(\ell^+ - 1) + (p_{i^-}q_{i^-}\ell^- + \ell^+)(\ell^- - 1) + 1.$$

So

$$\begin{aligned} \mu^+ - \mu^- &= -p_{i^-}q_{i^-}\ell^-(\ell^- - 1) + \sum_{i=1}^r [p_i(q_i k_i + \cdots + q_r k_r) + q_i(p_1 k_1 + \cdots + p_{i-1} k_{i-1})] k_i \\ &\quad - (p_1 k_1 + \cdots + p_r k_r), \end{aligned}$$

$$\begin{aligned} \mu^+ - \mu^- &= -p_{i^-}q_{i^-}\ell^-(\ell^- - 1) + \left(\sum_{i=1}^r p_i k_i \right) (\ell^- - 1) \\ &\quad + \sum_{i=1}^r [(p_1 q_i - p_i q_1) k_1 + \cdots + (p_{i-1} q_i - p_i q_{i-1}) k_{i-1}] k_i. \end{aligned}$$

From the equality

$$\sum_{i=1}^r p_i k_i = \frac{p_1}{q_1} \ell^- + \sum_{i=2}^r \frac{p_i q_1 - p_1 q_i}{q_1} k_i.$$

we get

$$\begin{aligned} \mu^+ - \mu^- &= \frac{p_1 - p_i^- q_i^- q_1}{q_1} \ell^- (\ell^- - 1) + \sum_{i=2}^r \left(\frac{\ell^- - 1}{q_1} (p_i q_1 - p_1 q_i) + \sum_{j=1}^{i-1} (p_j q_i - p_i q_j) k_j \right) k_i. \\ &= \frac{\Delta_0}{q_1} \ell^- (\ell^- - 1) + \sum_{i=2}^r \left(\frac{\ell^- - 1}{q_1} \Delta_{i,1} - \sum_{j=1}^{i-1} \Delta_{i,j} k_j \right) k_i. \end{aligned}$$

We also have

$$\Delta_{i,j} = \sum_{t=j}^{i-1} \frac{q_i q_j}{q_t q_{t+1}} \Delta_t$$

so

$$\sum_{j=1}^{i-1} \sum_{t=j}^{i-1} \frac{q_i q_j}{q_t q_{t+1}} \Delta_t k_j = q_i \sum_{t=1}^{i-1} \frac{\Delta_t}{q_t q_{t+1}} \sum_{j=1}^t q_j k_j.$$

We get the result from the equality:

$$\begin{aligned} \mu^+ - \mu^- &= \frac{\Delta_0}{q_1} \ell^- (\ell^- - 1) + \sum_{i=2}^r \sum_{t=1}^{i-1} q_i \frac{\Delta_t}{q_t q_{t+1}} (n_t - 1) k_i, \\ &= \frac{\Delta_0}{q_1} \ell^- (\ell^- - 1) + \sum_{t=1}^{r-1} \sum_{i=t+1}^r q_i \frac{\Delta_t}{q_t q_{t+1}} (n_t - 1) k_i. \end{aligned}$$

□

Lemma 7.5

$$\alpha^+ - \alpha^- = C_0^+ (\ell^+ - \ell^-).$$

Proof. From the definitions. □

From the previous lemmas, we deduce:

Proposition 7.6

$$\begin{aligned} (6S - \mu \alpha_\mu)_w &= - \left(\frac{d^2 - \ell^-}{(q_1 \ell^+ + p_1 \ell^-)(\ell^+ + p_i^- q_i^- \ell^-)} - \frac{\ell^- - 1}{q_1} (\ell^- - 1 - \ell^- \alpha_0) \right) \Delta_0 \\ &\quad - \sum_{t=1}^{r-1} \left(E_t^+ + \frac{n_t(n_t - 1)}{q_t q_{t+1}} (\alpha_0 - \alpha^+) \right) \Delta_t. \end{aligned}$$

Proof. From the previous Lemmas we have:

$$(6S - \mu\alpha_\mu)_w = -E_2^+ \Delta_2 - \dots - E_{r-1}^+ \Delta_{r-1} \\ + [d^2 - \ell^- + (\ell^+ + p_{i-}^- q_{i-}^- \ell^-)(\ell^+ - \ell^-)(\ell^- - 1)] (C_0^+ - C_0^-) \\ + (\mu^+ - \mu^-)(\alpha^+ - \alpha_0).$$

So we have immediately the coefficient of Δ_t for $t > 0$ and for Δ_0 we have to remark that

$$\alpha^+ \frac{\ell^-}{q_1} - \frac{\ell^+ - \ell^-}{q_1 \ell^+ + p_1 \ell^-} = \frac{\ell^- - 1}{q_1}.$$

□

Now it is easy to prove that the Hertling conjecture is true in dimension 2.

Theorem 7.7 *There exists positive rational numbers $(E_e)_{e \in \text{Ed}}$ such that*

$$6S - \mu\alpha_\mu = - \sum_{e \in \text{Ed}} E_e \Delta_e.$$

Proof. We just have to show that the first coefficient is positive.

We know that α_0 is greater than

$$\alpha_i = 1 - \frac{p_{i-}^- + q_{i-}^-}{\ell^+ + p_{i-}^- q_{i-}^- \ell^-}$$

so we only have to check the inequality for α_i instead of α_0 .

We have

$$q_1(d^2 - \ell^-) + (q_1 \ell^+ + p_1 \ell^-)(\ell^+ + p_{i-}^- q_{i-}^- \ell^- - \ell^-(p_{i-}^- + q_{i-}^-))(\ell^- - 1) = \\ (d^2 - 1)q_1 + [(q_1 \ell^+ + p_1 \ell^-)(\ell^+ + ((p_{i-}^- - 1)(q_{i-}^- - 1) - 1)\ell^-) - q_1] (\ell^- - 1)$$

If $p_{i-}^-, q_{i-}^- > 1$ then the last expression is positive. If p_{i-}^- or q_{i-}^- is equal to 1 then $\ell^+ > \ell^-$ then the last expression is also positive. □

Corollary 7.8 *If $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ is an isolated singularity then the conjecture of Hertling is true for f . Moreover, we have an equality if and only if f is a positive deformation of a quasi-homogeneous polynomial defining an isolated singularity.*

References

- [AGV] V.I. Arnold, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of differentiable maps*, vol 1,2, Birkhäuser, Boston, 1988.
- [B1] T. Brélivet, *Variance of the spectral number and Newton polygons*, Bull. Sci. Math. 126 (2002), no. 4, 332–342.
- [B2] T. Brélivet, *Topologie des polynômes, spectre et variance du spectre*, Thèse Université Bordeaux 1, 2002.
- [B3] T. Brélivet, *Sur les paires spectrales de polynômes à deux variables*, Preprint.
- [C] P. Cassou-Noguès, *Entrelacs toriques itérés et intégrales associées à une courbe plane*, Séminaire de Théorie des Nombres, Bordeaux 2 (1990), 273–331.
- [CP] P. Cassou-Noguès, A. Ploski, *Introduction to Algebraic Plane Curve Singularities*, in preparation.
- [Di] A. Dimca, *Monodromy and Hodge theory of regular functions*, Proceedings of the Summer Institute on Singularities, Newton Institute, Cambridge 2000.
- [EN] D. Eisenbud, W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Mathematics Studies, 110, Princeton University Press, 1985.
- [H] C. Hertling, *Frobenius manifolds and variance of the spectral numbers*, Proceedings of the Summer Institute on Singularities, Newton Institute, Cambridge 2000, see also Preprint math.CV/0007187.
- [KV] A. G. Khovanskiĭ, A. N. Varchenko, *Asymptotic behavior of integrals over vanishing cycles and the Newton polyhedron*, Dokl. Akad. Nauk SSSR 283 (1985), no. 3, 521–525.
- [N] W.D. Neumann, *Topology of hypersurface singularities*, Preprint. <http://www.math.columbia.edu/~neumann/preprints/kaehler1.ps>
- [S1] M. Saito, *Period mapping via Brieskorn modules*, Bull. Soc. math. France 119 (1991), 141–171.
- [S2] M. Saito, *Exponents of an irreducible plane curve singularity*, Preprint math.AG/ 0009133.
- [SS] J. Scherk and J. Steenbrink, *On the mixed Hodge structure on the cohomology of the Milnor fiber*, Math. Ann. 271 (1985), 641–665.

- [SSS] R. Schrauwen, J. Steenbrink, J. Stevens, *Spectral pairs and the topology of curves singularities*, Proceedings of Symposia in Pures Mathematics, Volume 53 (1991), 305-328.
- [St1] J. Steenbrink, *Mixed Hodge structures on the vanishing cohomology*, in P. Holm ed.: Real and Complex Singularities, Oslo 1976.
- [St2] J. Steenbrink, *Applications of Hodge theory to singularities*, Proc. I.C.M. Kyoto (1990), 559-576.
- [V] A. N. Varchenko, *The asymptotics of holomorphic forms determine a mixed Hodge structure*, Sov. Math. Dokl. 22(1980), 248-252).

Thomas Brélivet
 Departamento de Álgebra, Geometría y Topología
 Facultad de ciencias
 Universidad de Valladolid
 47005 Valladolid
 España
 email: brelivet@agt.uva.es